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Complemented Graphs and Blow-ups of Boolean Graphs, with Applications to Co-maximal Ideal Graphs

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Abstract. For a set X, let 2^X be the power set of X. Let B_X be the Boolean graph, which is defined on the vertex set $2^X \setminus \{X, \emptyset\}$, with M adjacent to N if $M \cap N = \emptyset$. In this paper, several purely graph-theoretic characterizations are provided for blow-ups of a finite or an infinite Boolean graph (respectively, a pre-atomic graph). Then the characterizations are used to study co-maximal ideal graphs that are blow-ups of Boolean graphs (pre-atomic graphs, respectively).

1. Introduction

Recall that a Boolean graph is defined to be the zero divisor graph $\Gamma(R)$ of a Boolean ring R, see [12, 19] (see also [3, 5, 9, 20]). Recall that a finite Boolean graph is isomorphic to $B_n = \Gamma(\prod_{i=1}^{n} \mathbb{Z}_2)$ for some positive integer n. Note that B_n is isomorphic to the zero divisor graph of the finite semilattice $(2^{[n]}, \cap)$, where $[n]$ denotes the set $\{1, 2, \ldots, n\}$ and $2^{[n]}$ is the power set of $[n]$ throughout the paper. For a general nonempty set X, we use B_X to denote the zero divisor graph of the meet-semilattice $(2^X, \cap)$, i.e., the vertex set of B_X is $2^X \setminus \{X, \emptyset\}$, with distinct $M, N \subseteq X$ adjacent if and only if $M \cap N = \emptyset$. Clearly, $B_{[n]} = B_n$. Throughout the paper, let S be the subgraph of B_X induced on $\{\{x\} \mid x \in X\}$. Then S is the unique maximum clique of B_X (see Definition 2.1 for the definition of a maximum clique when $|X| = \infty$). B_X is also denoted as B_S.

All graphs in the paper are assumed to be undirected and simple. For a graph G, the vertex set of G is denoted by $V(G)$. For a vertex $v \in V(G)$, the neighborhood of v, denoted by $N(v) = \{u \in V(G) \mid u \sim v\}$, is the set of all vertices adjacent to v in the graph G. For a subgraph A of G, denote $N(A) = \{N(v) \mid v \in V(A)\}$. For other concepts and notations in graph theory, we use [18] as a basic reference.

Blow-up is an interesting technique in graph theory. Roughly speaking, to blow-up a graph G is to replace every vertex x of G by a set T_x to get a possibly new and larger graph G_T, where $|T_x| \geq 1$. The induced subgraph of G_T on T_x is a discrete graph, i.e., a graph without any edge, while for distinct vertices x, y of G, each vertex of T_x is adjacent to all vertices of T_y in G_T if and only if x is adjacent to y in G, see [8, 16, 17] for details. The previous work shows that graph blow-up plays an essential role in the theory of the co-maximal ideal graph of a ring, see [21, 22] for the concise definitions, the history, the recent development, and a list of references.

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This paper is organized in the following way. In Section 2, some new definitions are introduced, and some characterizations are established for blow-ups of Boolean graphs with a finite or an infinite maximum clique. In Section 3, conditions M, N and N^* are introduced, which are closely related to neighbourhoods, and the relationship among them are studied. Complemented graphs are studied in Section 4, and an additional characterization about a Boolean graph and its blow-up is given in Section 5 by taking advantage of the conditions established in Section 3. In Section 6, applying the characterization of a blow-up of a Boolean graph to the co-maximal ideal graph of a commutative ring, a new alternative proof to the main theorem in [21] is given.

2. Characterizations of Boolean graphs, pre-atomic graphs and their blow-ups

In this section, we are going to characterize a blow-up of a Boolean graph. We start with a concise definition for a (possibly infinite) maximum clique in a graph.

Definition 2.1. *A clique S of a graph G is called a maximum clique of G if the following conditions are satisfied:*
(1) *$|V(S)|$ is maximal in $\{|V(L)| \mid L$ is a clique of $G\}$.*
(2) *For any finite subset $A \subseteq V(S)$ and subset $B \subseteq V(G) \setminus V(S)$ with $|B| = |A| + 1$, the subgraph induced on $B \cup (V(S) \setminus A)$ is not a clique of the graph G.*

Note that if S is a maximum clique of G, then there is no clique properly containing S. In fact, it follows from condition (2) when A is taken to be an empty set.

For a graph with a finite clique number, a maximum clique is clearly a clique with the maximal number of vertices. But for a graph with an infinite clique, there may be other definitions for a maximum clique.

For later usage in this section as well as next sections, we begin with a characterization of a Boolean graph B_S. For a set X, we call the subset A of X to be nontrivial, if $A \neq \emptyset$ and $A \neq X$. In order to simplify the notation, we sometimes use x to denote the subset $\{x\}$ in 2^X.

Theorem 2.2. *Let G be a graph with a maximum clique S. Then G is isomorphic to the Boolean graph B_S if and only if the following properties are satisfied:*
(1) *For each vertex $v \in V(G)$, $N(v) \cap V(S)$ is a nontrivial subset of $V(S)$; For each nontrivial subset A of $V(S)$, there exists a vertex $v \in V(G)$ such that $A = N(v) \cap V(S)$.*
(2) *G is uniquely $S \cap N$-determined* (or alternatively, *G* is uniquely *N*-determined), *i.e., $V(S) \cap N(x) = V(S) \cap N(y)$* (respectively, *$N(x) = N(y)$*) *implies $x = y$ for vertices $x, y \in V(G)$.*
(3) *For vertices $x, y \in V(G)$, $V(S) \subseteq N(x) \cup N(y)$ holds if and only if x is in $N(y)$.*

Proof. ($\Longrightarrow$) Assume that $G = B_S = B_X$, where S is a maximum clique of G with $V(S) = \{\{x\} \mid x \in X\}$.

(1) Since each vertex $v \in V(G)$ is a nontrivial subset of X, it follows that

$$N(v) \cap V(S) = \{\{t\} \mid t \in X \setminus v\}$$

is a nontrivial subset of $V(S)$. For each nontrivial subset A of $V(S)$, take $v = \{t \mid \{t\} \in V(S) \setminus A\} \in 2^X$ and then it is easy to see that $N(v) \cap V(S) = A$.

(2) It follows from [12] that G is uniquely N-determined, i.e., $N(x) = N(y)$ implies $x = y$ for vertices x, y of B_S (*i.e.*, nontrivial $x, y \subseteq X$). Thus it is only necessary to check that $V(S) \cap N(x) = V(S) \cap N(y)$ implies $N(x) = N(y)$. Assume to the contrary that $N(x) \nsubseteq N(y)$ holds for a pair of $x, y \subseteq X$. Then there exists $k \in X$ such that $k \in y \setminus x$. Hence $\{k\} \in N(x) \setminus N(y)$, contradicting $V(S) \cap N(x) = V(S) \cap N(y)$.

(3) For vertices $x, y \in V(G)$, assume first $x \in N(y)$. Then $x \cap y = \emptyset$. For any $\{k\} \in V(S)$, if $\{k\} \notin N(x)$, then $\{k\} \subseteq x$, hence $\{k\} \nsubseteq y$. Then $\{k\} \in N(y)$. This shows $V(S) \subseteq N(x) \cup N(y)$. Conversely, if $x \notin N(y)$, then $x \cap y \neq \emptyset$. Assume $k \in x \cap y$. Then $\{k\} \notin N(x) \cup N(y)$, hence it implies $V(S) \nsubseteq N(x) \cup N(y)$.

($\Longleftarrow$) Assume that (1) and (3) hold. Assume further that G is N-determined. For a vertex $v \in V(G)$, denote $B(v) = \{u \in V(S) \mid u \notin N(v)\}$. It is clear that $\varphi : v \to B(v)$ is a map from $V(G)$ to $V(B_S) = 2^{V(S)} \setminus \{V(S), \emptyset\}$.

By condition (1), the map is surjective. Assume $B(u) = B(v)$. Then by the definition of $B(u)$, it follows that $V(S) \cap N(u) = V(S) \cap N(v)$. Then by condition (3), we have

$$z \in N(u) \Longleftrightarrow V(S) \subseteq N(u) \cup N(z) \Longleftrightarrow V(S) \subseteq N(v) \cup N(z) \Longleftrightarrow z \in N(v),$$

so $N(u) = N(v)$ holds. Then $u = v$ since G is assumed to be N-determined. This shows that φ is also injective and thus bijective.

In the following, we prove that $x \sim y$ in G if and only if $\varphi(x) \sim \varphi(y)$ in B_S. Assume first that $x \sim y$ in G. Then $x \in N(y)$, thus by condition (3), $V(S) \subseteq N(x) \cup N(y)$. To check that $B(x) \sim B(y)$ in B_S is equivalent to verifying $B(x) \cap B(y) = \emptyset$. In fact, for any $u \in V(S)$, $u \notin B(x) \cap B(y)$ is equivalent to saying either $u \in N(x)$ or $u \in N(y)$. Thus $x \sim y$ implies $B(x) \sim B(y)$. Conversely, if $B(x) \sim B(y)$ in B_S, then clearly $x \sim y$ in G by the above argument. This shows that φ induces a graph isomorphism from G to B_S and it completes the proof. □

Note that under the assumption (3), the equality $V(S) \cap N(x) = V(S) \cap N(y)$ is equivalent to the equality $N(x) = N(y)$. Note also that if G has a maximum clique and conditions (2) and (3) hold, then G has a unique maximum clique, as the following proposition shows.

Proposition 2.3. *Let G be a graph with a maximum clique S, where $|V(S)|$ is either a finite number larger than 1 or an infinite cardinal number. If conditions* (2) *and* (3) *of Theorem 2.2 are satisfied for G relative to S, then S is the unique maximum clique of G.*

Proof. If U is a maximum clique of G, we will show that $U = S$. First, we claim that for each $u \in V(U)$, there exists one and only one $v \in V(S)$, such that $v \notin N(u)$. In fact, $V(S) \setminus N(u) \neq \emptyset$ since S is a maximum clique of G. Assume to the contrary that $V(S) \setminus N(u)$ contains more than one elements of $V(S)$, and assume without loss of generality that $v_1, v_2 \in V(S) \setminus N(u)$. Clearly, $|V(U)| \geq 2$. Let $u_1 = u$, then there exists a $u_2 \in V(U) \setminus \{u_1\}$. Since $u_1 \in N(u_2)$, it follows from condition (3) that $\{v_1, v_2\} \subseteq N(u_2)$. In a similar way, we have $\{v_1, v_2\} \subseteq N(w)$ for every $w \in V(U) \setminus \{u_1\}$, thus $\{v_1, v_2\} \cup (V(U) \setminus \{u_1\})$ is a clique in G, contradicting the assumption on U. This shows that for each $u \in V(U)$, there exists only one $v \in V(S) \setminus N(u)$.

Then $V(S) \cap N(u) = V(S) \setminus \{v\} = V(S) \cap N(v)$. By condition (2) of Theorem 2.2 we have $u = v$. Thus $V(U) \subseteq V(S)$ and hence $U = S$. This completes the proof. □

In the following, we will show that the first part of condition (1) in Theorem 2.2 can be replaced by a condition " G is connected". We need the following lemma.

Lemma 2.4. *Let G be a graph with a clique S. If $N(v) \cap V(S)$ is a nontrivial subset of $V(S)$ for each vertex $v \in V(G)$, then G is connected.*

Proof. Note that any vertex of G is adjacent to at least one vertex of S, and each pair of vertices in S is connected since S is a clique. So, it is clear that G is connected. □

Proposition 2.5. *In Theorem 2.2, the first part of condition* (1) *can be replaced by the condition " G is connected ".*

Proof. By Lemma 2.4, it suffices to show that $N(v) \cap V(S)$ is a nontrivial subset of $V(S)$ for each vertex $v \in V(G)$, when G is connected. As in the proof of Theorem 2.2, define a map $\varphi : v \to B(v)$ from $V(G)$ to $V(B_S) = 2^{V(S)} \setminus \{V(S), \emptyset\}$. In fact, $B(v)$ is not empty for each $v \in V(G)$ since S is a maximum clique of G. On the other hand, if $B(v) = V(S)$ holds for some $v \in V(G)$, then $V(S) \cap N(v) = \emptyset$. Since G is assumed to be connected, there exists a vertex $u \in N(v)$. Then condition (3) implies $V(S) \subseteq N(u)$, contradicting the assumption on S. □

From the proof, one knows that if condition (3) of Theorem 2.2 is satisfied for a graph, then the connectivity of the graph is equivalent to the first part of condition (1).

Now we use Theorem 2.2 to characterize a finite or an infinite blow-up of a Boolean graph:

Theorem 2.6. *Let G be a graph with a maximum clique S. Then G is a graph blow-up of the Boolean graph B_S if and only if the following properties are satisfied:*
(1) *For each vertex $v \in V(G)$, $N(v) \cap V(S)$ is a nontrivial subset of $V(S)$; For each nontrivial subset $A \subseteq V(S)$, there exists a vertex $v \in V(G)$ such that $N(v) \cap V(S) = A$.*
(2) *For vertices $x, y \in V(G)$, $V(S) \subseteq N(x) \cup N(y)$ holds if and only if $x \in N(y)$.*

Proof. ($\Longrightarrow$) If G is a graph blow-up of B_S, then B_S is a retract of G. By Theorem 2.2 and the definition of graph blow-up, both (1) and (2) hold.

($\Longleftarrow$) Assume that conditions (1) and (2) hold for a graph G. Define an equivalence relation in $V(G)$ by the following:

x is equivalent to y if and only if $N_G(x) = N_G(y)$.

Then we proceed to define a new graph $\overline{G}$: First, let $V(\overline{G})$ be the set of equivalent classes under the relation. Then, for distinct $\overline{u}$ and $\overline{v}$ in $V(\overline{G})$, define $\overline{u} \sim \overline{v}$ iff $u \in N_G(v)$ in the graph G. We claim that the edge is well-defined, i.e. $\overline{u} \sim \overline{v}$ is independent of the choice of u and v. In fact, if $\overline{u_1} = \overline{u_2}$, $\overline{v_1} = \overline{v_2}$, and $\overline{u_1} \sim \overline{v_1}$, then $u_1 \sim v_1$ in G, thus $V(S) \subseteq N_G(u_1) \cup N_G(v_1) = N_G(u_2) \cup N_G(v_2)$. So, $u_2 \sim v_2$ in G. For distinct $x, y \in V(S)$, condition (2) implies $N_G(x) \neq N_G(y)$, so that $|V(S)| = |V(\overline{S})|$, where $V(\overline{S}) = \{\overline{v} \mid v \in V(S)\}$. Thus the graph $\overline{G}$ has a maximum clique $\overline{S}$. By assumption, conditions (1) to (3) of Theorem 2.2 are clearly satisfied for the newly defined graph $\overline{G}$, thus $\overline{G}$ is a Boolean graph. In the following, we show that G is a blow-up of $\overline{G}$.

For any $\overline{x} \in V(\overline{G})$, let $T_{\overline{x}} = \{y \in V(G) \mid N_G(y) = N_G(x)\}$. Clearly, $y_1 \notin N_G(y_2)$ holds for any $y_1, y_2 \in T_x$, i.e., no two vertices in a T_x are adjacent in G. Furthermore, for any $\overline{x}, \overline{y} \in V(\overline{G})$, if $\overline{x} \in N_{\overline{G}}(\overline{y})$, then $x \in N_G(y)$. Then for any $x_1 \in T_x$ and $y_1 \in T_y$, we have $x \in N_G(y) = N_G(y_1)$, thus $y_1 \in N_G(x) = N_G(x_1)$. This means that each vertex in T_x is adjacent to every vertex in T_y. On the other hand, if $\overline{x}$ is not adjacent to $\overline{y}$ in $\overline{G}$, then clearly no vertex of T_x is adjacent to a vertex in T_y. This shows that G is a blow-up of the Boolean graph $\overline{G}$.
□

In the following, we study a special class of subgraphs of B_S induced on a nonempty subset of $V(B_S)$, where $V(S)$ is a finite or an infinite set.

Definition 2.7. A *pre-atomic graph A_S* is an induced subgraph of B_S, such that A_S contains the unique maximum clique S of B_S.

It follows from the definitions of the Boolean graph B_S and the pre-atomic graph A_S that, for each pair of $M, N \in V(A_S)$, $M \sim N$ if and only if $M \cap N = \emptyset$.

As in the proof of Lemma 2.4, a pre-atomic graph is always connected with diameter less than four. Whenever $|V(S)| \geq 3$, the girth of a pre-atomic graph is three. Also, a Boolean graph is a pre-atomic graph, but the converse is clearly not true. The following is compared with Theorem 2.2, and the proof is omitted.

Proposition 2.8. *For a graph G, G is isomorphic to a pre-atomic graph A_S if and only if in G there exists a maximum clique K such that $|V(K)| = |V(S)|$ and the following properties are satisfied:*
(1) *For each vertex $v \in V(G)$, $N(v) \cap V(K)$ is a nontrivial subset of $V(K)$;*
(2) *G is uniquely $K \cap N$-determined, i.e., $V(K) \cap N(x) = V(K) \cap N(y)$ implies $x = y$ for vertices $x, y \in V(G)$;*
(3) *For vertices $x, y \in V(G)$, $V(K) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

We observe that a pre-atomic graph has a unique maximum clique in view of Proposition 2.3. Since the proof of the following proposition is similar to that of Theorem 2.6, we omit it here.

Proposition 2.9. *A graph G is isomorphic to a blow-up of a pre-atomic graph A_S if and only if in G there exists a maximum clique K such that $|V(K)| = |V(S)|$ and the following properties are satisfied:*
(1) *For each vertex $v \in V(G)$, $N(v) \cap V(K)$ is a nontrivial subset of $V(K)$;*
(2) *For vertices $x, y \in V(G)$, $V(K) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

Note that both Theorem 2.6 and Proposition 2.9 only refer to one maximum clique of graph G. In fact, if one maximum clique possesses the properties described in these propositions, so do the other maximum cliques.

By Proposition 2.9 and Theorem 4.4 in [10], the following corollary is clear.

Corollary 2.10. *For a connected graph G, it is a blow-up of a pre-atomic graph if and only if it is the zero-divisor graph of an atomic poset.*

A pre-atomic graph G is called an *atomic graph*, if the following $\overline{N}$-condition is satisfied:

$\overline{N}$**-condition**: *For each pair of vertices $x, y \in V(G)$, $x \notin N(y)$ implies that there exists a vertex $z \in V(G)$ such that $N(x) \cup N(y) \subseteq \overline{N(z)}$, where $\overline{N(z)} = N(z) \cup \{z\}$.*

By [6, Theorem 1(4)], each zero divisor graph of a commutative semigroup satisfies the $\overline{N}$- condition. Thus for any zero divisor graph $G = \Gamma(T)$ of a commutative semigroup T, if G is pre-atomic, then it is an atomic graph.

As in the proof of Theorem 2.2, define the map $\varphi : v \to B(v)$ from an atomic graph $V(G)$ to $V(B_X)$ for some set X. It is clear that φ is injective. But even if one adds a least element 0 and a largest element 1 to $Im(\varphi)$, the resulting subset of $V(B_X)$ may be not a semilattice under the order relation of inclusion, as the following example shows:

Example 2.11. Consider the subgraph of the Boolean graph B_4 induced on the vertex set

$$V(G) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}\}.$$

The graph G is the complete graph K_4 together with three end vertices adjacent to three vertices of K_4 respectively. Clearly, it is an atomic graph. However, it follows from [14, Theorem 2.2] that G is not the zero divisor graph of any semigroup.

3. Conditions *M*, *N* and *N**

For a graph G and a vertex u, if for each $v \in V(G)$, $N(u) \subseteq N(v)$ implies $N(u) = N(v)$, then $N(u)$ is called a *maximal neighbourhood* in $N(G)$. Let $Max(N(G))$ be the set of all maximal neighbourhoods in $N(G)$. We call a graph G satisfying the *neighbourhood* condition (abbreviated as the *N-condition*), if for each pair of nonadjacent vertices $u, v \in V(G)$, there exists a vertex w such that $N(u) \cup N(v) \subseteq N(w)$. We call a graph G satisfying the *N^*-condition*, if G has a maximum clique S, and for each pair of nonadjacent vertices $u, v \in V(G)$, there exists a vertex w such that $N(u) \cup N(v) \subseteq N(w)$ and

$$V(S) \cap (N(u) \cup N(v)) = V(S) \cap N(w).$$

In [13], a graph G is called a *compact* graph if G contains no isolated vertex and it satisfies the N-condition. It is also proved in [13] that G is a compact graph if and only if G is the zero divisor graph of a poset. For further details on compact graphs, one can refer to [13] and the included references.

Lemma 3.1. *Let G be a finite or an infinite graph satisfying the N-condition. Assume that G has a maximum clique. Then the following statements hold:*

(1) *Denote $C(x) = \{y \mid N(y) = N(x)\}$. If S is a maximum clique of G, then any T, such that $V(T)$ constructed by choosing one and only one vertex of $C(x)$ for each $x \in V(S)$, is a maximum clique of G.*

(2) *S is a maximum clique of G if and only if $N(S) = Max(N(G))$ and $N(u) \neq N(v)$ holds for distinct $u, v \in V(S)$.*

Proof. (1) Let φ be a bijection from $V(S)$ to $V(T)$, such that $\varphi(x) \in C(x)$ for each $x \in V(S)$, hence $N(x) = N(\varphi(x))$. Clearly, T is a clique since S is a clique. If T is not maximum, by Definition 2.1, one of the following three cases is satisfied.

Case 1, there exists a clique K, such that $|V(T)| < |V(K)|$. Note that $|V(S)| = |V(T)| < |V(K)|$, a contradiction.

Case 2, there exists a clique K, such that $V(T) \subsetneq V(K)$. Then there exists $x \in V(K) \setminus V(T)$, such that $x \in N(y)$ for each $y \in V(T)$. So, $x \in N(\varphi^{-1}(y))$ for each $y \in V(T)$, i.e., $x \in N(z)$ for each $z \in V(S)$. So, $\{x\} \cup V(S)$ induces a clique, a contradiction.

Case 3, there exists a nonempty finite subset $A \subseteq V(T)$ and $B \subseteq V(G) \setminus V(T)$ where $|B| = |A| + 1$, such that $B \cup (V(T) \setminus A)$ induces a clique. In a similar way as case 2, it is not hard to see that $B \cap (V(S) \setminus \varphi^{-1}(A)) = \emptyset$ and $B \cup (V(S) \setminus \varphi^{-1}(A))$ induces a clique, a contradiction.

(2) ($\Longrightarrow$) Assume that S is a maximum clique of G and let $v_1 \in V(S)$. If $N(v_1) \notin Max(N(G))$, then there exists a vertex $v \in V(G)$, such that $N(v_1) \subsetneq N(v)$. Then take a vertex $u \in V(G)$, such that $u \in N(v) \setminus N(v_1)$. Since G satisfies the N-condition, there exists a vertex $w \in V(G)$, such that $N(u) \cup N(v_1) \subseteq N(w)$. Then $v \in N(w)$ and $N(v_1) \subseteq N(w)$, thus $\{v, w\} \subseteq V(G) \setminus V(S)$ and $\{v, w\} \cup (V(S) \setminus \{v_1\})$ induces a clique in G, contradicting assumption on S (see condition (2) in Definition 2.1). The contradiction shows that $N(v_1) \in Max(N(G))$ for each $v_1 \in V(S)$. For each pair of distinct vertices u, v in $V(S)$, $N(u) \neq N(v)$ clearly follows from $u \sim v$ in G. Thus $N(S)$ is a subset of $N(G)$ consisting of mutually distinct maximal neighbourhoods in $N(G)$.

If $N(S) \subsetneq Max(N(G))$, then there is a vertex $z \in V(G) \setminus V(S)$, such that $N(z)$ is maximal in $N(G)$ and $N(z) \neq N(s)$ for any $s \in V(S)$. We claim that the subgraph induced on $V(S) \cup \{z\}$ is a clique of G, contradicting assumption on S (see condition (2) in Definition 2.1). In fact, if there exists a pair of distinct vertices v, u in $V(S) \cup \{z\}$ such that $v \notin N(u)$, then there exists $w \in V(G)$ such that $N(v) \cup N(u) \subseteq N(w)$. Then $N(v) \subsetneq N(w)$, contradicting the maximality of $N(v)$ in $N(G)$. In conclusion, $N(S)$ consists of all maximal neighbourhoods in $N(G)$, and $N(u) \neq N(v)$ holds for distinct $u, v \in V(S)$.

($\Longleftarrow$) Let T be a maximum clique of G. By the necessity, $N(T) = N(S)$ consists of all maximal neighbourhoods in $N(G)$ and $N(u) \neq N(v)$ holds for distinct $u, v \in V(T)$. Hence there exists a bijection φ from T to S, such that $N(x) = N(\varphi(x))$ for each $x \in V(T)$. By (1), S is a maximum clique of G. □

The following example shows that the above lemma is true when $\omega(G) = 1$.

Example 3.2. Let G be a discrete graph. For any $v \in V(G)$, $\{v\}$ induces a maximum clique of G. Since $N(v) = \emptyset$ for each $v \in V(G)$, $Max(N(G)) = \{\emptyset\}$. In this case, the above lemma holds clearly.

The following corollary adds something new to the compact graphs:

Corollary 3.3. *Let G be a compact graph (i.e., the zero divisor graph of a poset), and assume that G has a maximum clique. Then for any induced subgraph S of G, S is a maximum clique of G if and only if $N(S) = Max(N(G))$ and $N(u) \neq N(v)$ holds for distinct $u, v \in V(S)$.*

Corollary 3.4. *Let G be a graph satisfying the N-condition. Assume that S is a maximum clique of G. Then for a vertex $v \in V(G)$ and a vertex $u \in V(S)$, either $u \in N(v)$ or $N(v) \subseteq N(u)$ holds.*

Proof. If $u \notin N(v)$, then $N(u) \cup N(v) \subseteq N(w)$ holds for some vertex $w \in V(G)$. By Lemma 3.1, $N(u)$ is a maximal neighbourhood in $N(G)$, thus $N(w) = N(u)$ and hence $N(v) \subseteq N(u)$. □

Note that the following Lemma 3.5 is proved in [10], and Lemma 3.7 in [13]. We include a proof to each lemma for reader's convenience.

Lemma 3.5. *([10, Lemma 2.1]) If a graph G has a maximum clique S and satisfies the N-condition, then for any pair of distinct vertices $x, y \in V(G)$, $x \in N(y)$ if and only if $V(S) \subseteq N(x) \cup N(y)$.*

Proof. For each pair $x, y \in V(G)$, if $V(S) \subseteq N(x) \cup N(y)$, then we claim $x \in N(y)$, since otherwise, $x \notin N(y)$ and then there exists $z \in V(G)$, such that $N(z) \supseteq N(x) \cup N(y) \supseteq V(S)$. So, $\{z\} \cup V(S)$ induces a clique properly containing S, contradicting assumption on S. If $V(S) \nsubseteq N(x) \cup N(y)$, then there exists $v_i \in V(S)$ such that

$v_i \notin N(x) \cup N(y)$. By Corollary 3.4, $N(x) \cup N(y) \subseteq N(v_i)$ holds and then it is easy to see that $x \notin N(y)$ also holds: In fact, if $x \in N(y)$, then $x \in N(v_i)$, a contradiction. □

By Proposition 2.9, the following corollary is clear.

Corollary 3.6. *If G is a compact graph, with a maximum clique S, then G is a blow-up of a pre-atomic graph.*

Proof. By Proposition 2.9 and Lemma 3.5, it is sufficient to show that for each $v \in V(G)$, $N(v) \cap V(S)$ is a nontrivial subset of $V(S)$. Since S is a maximum clique of G, so $V(S) \nsubseteq N(v)$ and $\cap_{v_i \in V(S)} N(v_i) = \emptyset$. It suffices to show that $N(v) \cap V(S) \neq \emptyset$. If $v \nsim v_i$ for each $v_i \in V(S)$, then $N(v) \subseteq N(v_i)$ for each $v_i \in V(S)$ by Corollary 3.4. Hence $N(v) \subseteq \cap_{v_i \in V(S)} N(v_i) = \emptyset$. So, v is an isolated vertex. It contradicts to the definition of a compact graph. □

Lemma 3.7. *([13, Lemma 2.5]) If a graph G, with finite clique number, satisfies the N-condition, then $N(G)$ satisfies the ACC condition (i.e., for a series of $x_i \in V(G)$, if $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_i) \subseteq \cdots$, then there exists some $k \geq 1$, such that $N(x_i) = N(x_k)$ while $i \geq k$).*

Proof. If there is a series of $x_i \in V(G)$, such that $N(x_1) \subsetneq N(x_2) \subsetneq \cdots \subsetneq N(x_i) \subsetneq \cdots$. Then there exists $u_1 \in N(x_2) \setminus N(x_1)$. Since G satisfies the N-condition, there exists $w_1 \in V(G)$ such that $N(x_1) \cup N(u_1) \subseteq N(w_1)$, hence $\{w_1, x_2\}$ is a clique. In a similar way, there exists $u_2 \in N(x_3) \setminus N(x_2)$ and $w_2 \in V(G)$, such that $N(x_2) \cup N(u_2) \subseteq N(w_2)$ and hence $\{w_1, w_2, x_3\}$ is a clique. By induction, there exists a clique of size n in G for any positive integer n, it is a contradiction. □

In the following sections, we will give a new characterization of Boolean graphs and, blow-ups of a Boolean graph respectively. In order to do this, we introduce a new condition:

Definition 3.8. *Let G be a graph with a maximum clique. We call a graph G satisfying the M-condition, if for a maximum clique S of G and each induced discrete subgraph D of G with $V(S) \nsubseteq \cup_{x \in V(D)} N(x)$, there exists a vertex $z \in V(G)$, such that the followings are satisfied:*
(1) $\cup_{x \in V(D)} N(x) \subseteq N(z)$;
(2) $V(S) \cap (\cup_{x \in V(D)} N(x)) = V(S) \cap N(z)$.

For a graph G with a maximum clique, the N-condition is independent with the M-condition. The following example shows that the M-condition does not imply the N-condition.

Example 3.9. Let G be the graph in the following figure. It is easy to check that G satisfies the M-condition, but the N-condition fails for the nonadjacent vertices v_1, v_6.

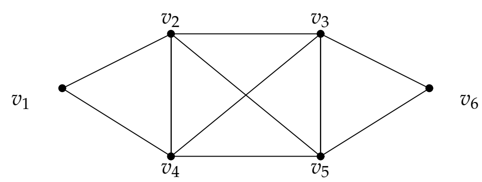

Figure 1

Lemma 3.10. *If a graph G, with a maximum clique S, satisfies the M-condition, and for each pair of vertices $x, y \in V(G)$, $V(S) \subseteq N(x) \cup N(y)$ implies $x \in N(y)$. Then G satisfies the N^*-condition.*

Proof. Assume that G satisfies the M-condition. For each pair of distinct vertices $x, y \in V(G)$, if $x \notin N(y)$, then $V(S) \nsubseteq N(x) \cup N(y)$. Let A be the discrete subgraph induced by $\{x, y\}$. Then there exists $z \in V(G)$, such that $N(x) \cup N(y) \subseteq N(z)$ and $V(S) \cap (N(x) \cup N(y)) = V(S) \cap N(z)$. Hence G satisfies the N^*-condition. □

Proposition 3.11. *Let G be a graph which has a finite clique number $\omega(G)$. Then G satisfies the N^*-condition if and only if G satisfies the conditions M and N.*

Proof. ($\Longleftarrow$) It is clear by Lemma 3.5 and Lemma 3.10.

($\Longrightarrow$) Assume that G satisfies the N^*-condition. It is clear that G satisfies the N-condition. In the following, we verify that G satisfies the M-condition. In fact, let S be a maximum clique of G. For an induced discrete subgraph A, if $V(S) \nsubseteq \cup_{x\in V(A)} N(x)$, then we claim that there exists a $z \in V(G)$, such that

$$\cup_{x\in V(A)} N(x) \subseteq N(z)$$

and

$$V(S) \cap (\cup_{x\in V(A)} N(x)) = V(S) \cap N(z).$$

Since $|V(S)| < \infty$, $V(A)$ can be divided into a finite number of mutually disjoint parts, denoted by $V(A) = \cup_{i\in\Gamma} V(A_i)$, such that for each pair of $x, y \in V(A_i)$, $V(S) \cap N(x) = V(S) \cap N(y)$. By Lemma 3.7 and Zorn's lemma, for each $i \in \Gamma$, there exists $z_i \in V(A_i)$ such that $N(z_i)$ is maximal in $N(A_i)$. Furthermore, it follows easily from the N^*-condition that $N(z_i)$ is actually the largest element in $N(A_i)$. So, for each $i \in \Gamma$, there exists $z_i \in V(A_i)$ such that

$$\cup_{x\in V(A_i)} N(x) = N(z_i), \;\; V(S) \cap (\cup_{x\in V(A_i)} N(x)) = V(S) \cap N(z_i).$$

In the following, we will complete the proof by induction on $|\Gamma|$. If $|\Gamma| = 2$, note that $z_1 \notin N(z_2)$, there exists $z \in V(G)$, such that

$$N(z_1) \cup N(z_2) \subseteq N(z), \;\; V(S) \cap (N(z_1) \cup N(z_2)) = V(S) \cap N(z).$$

Clearly, $\cup_{x\in V(A)} N(x) \subseteq N(z)$ and $V(S) \cap (\cup_{x\in V(A)} N(x)) = V(S) \cap N(z)$. Assume that the conclusion is proved when $|\Gamma| = n - 1$. Then for the case $|\Gamma| = n$, there exists $u \in V(G)$, such that

$$\cup_{i\in\Gamma, i\neq n} N(z_i) \subseteq N(u)$$

and

$$V(S) \cap (\cup_{i\in\Gamma, i\neq n} N(z_i)) = V(S) \cap N(u).$$

Since $V(S) \nsubseteq \cup_{x\in V(A)} N(x)$, $u \notin N(z_n)$ by Lemma 3.5. So, there exists $z \in V(G)$ such that $N(u) \cup N(z_n) \subseteq N(z)$ and $V(S) \cap (N(u) \cup N(z_n)) = V(S) \cap N(z)$. Clearly, $\cup_{x\in V(A)} N(x) \subseteq N(z)$ and

$$V(S) \cap (\cup_{x\in V(A)} N(x)) = V(S) \cap N(z)$$

hold. This completes the proof. □

4. Complemented graph

Recall from [5] that in a graph G, a vertex $v \in V(G)$ is called a complement of a vertex w, denoted by $v \perp w$, if v is adjacent to w, and no vertex is adjacent to both v and w. Clearly, if $v \perp w$, then there exists no triangle which contains vw as an edge. A graph G is called *complemented* if every vertex of G has a complement. Recall from [5] that a complemented graph G is said to be *uniquely complemented*, if $a \perp b$ and $a \perp c$ implies $N(b) = N(c)$. In the rest of this paper, we call a graph G to be *strongly complemented*, if G is complemented, and every vertex of G has a unique complement. It is clear that for a strongly complemented graph G, $N(a) \neq N(b)$ holds for each pair of distinct vertices $a, b \in V(G)$.

Lemma 4.1. *Let G be a graph satisfying the N-condition, and let S be a maximum clique of G. For each $x \in V(S)$, if $y \perp x$, then $N(z) = N(x)$ holds for each $z \in N(y)$.*

Proof. It follows from $y \perp x$ and $z \in N(y)$ that $z \notin N(x)$. By Corollary 3.4, $N(z) \subseteq N(x)$. If $N(z) \subsetneq N(x)$, then there exists $u \in N(x) \setminus N(z)$. Since G satisfies the N-condition, there exists $v \in V(G)$, such that $N(z) \cup N(u) \subseteq N(v)$. Because $z \in N(y)$, so $y \in N(v)$. Since $x \in N(u) \subseteq N(v)$, $v \in N(x) \cap N(y)$, a contradiction. □

Corollary 4.2. *Let G be a strongly complemented graph with a maximum clique S. If G satisfies the N-condition, then the number of the end vertices is identical with $\omega(G)$.*

Proof. First, we claim that for each $x \in V(S)$, the unique complement of x is an end vertex. By Lemma 4.1, if S is a maximum clique of G, then for each $x \in V(S)$, $y \perp x$ implies $N(z) = N(x)$ for each $z \in N(y)$. Because G is strongly complemented, so $z = x$, and hence y is an end vertex. In the following, we will show that for an end vertex u, the unique complement of u has a maximal neighbourhood. Actually, if $v \perp u$, then $u \in N(v)$. Since u is an end vertex, there is no neighbourhood properly containing $N(v)$. By Lemma 3.1, the proof is completed. □

Recall that for a graph G, the complement graph $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$, and for each $x, y \in V(\overline{G})$, $x \sim y$ in $\overline{G}$ if and only if $x \nsim y$ in G.

Corollary 4.3. *Let G be a complemented graph satisfying the N-condition. If $\omega(G) \geq 3$, then $\omega(G) \leq \omega(\overline{G})$, where $\overline{G}$ is the complement graph of G.*

Proof. Let S be a maximum clique of G. Since G is complemented, for each $x \in S$, there exists a vertex $\overline{x} \in V(G)$ such that $\overline{x} \perp x$. By Lemma 3.1 and Lemma 4.1, $\overline{x} \nsim \overline{y}$ if $x \neq y$ for $x, y \in V(S)$. So, $\{\overline{x} \mid x \in V(S)\}$ induces a discrete subgraph of G, i.e., $\{\overline{x} \mid x \in V(S)\}$ induces a clique in $\overline{G}$. □

In general, under the assumption of Corollary 4.3, $\omega(\overline{G})$ may larger than $\omega(G)$, as the following example shows.

Example 4.4. The following graph G is a blow-up of B_3, which is complemented and satisfies the N-condition as the following Corollary 5.2 shows. It is easy to see that $\omega(G) = \omega(B_3) = 3$, but $\omega(\overline{G}) = 4$.

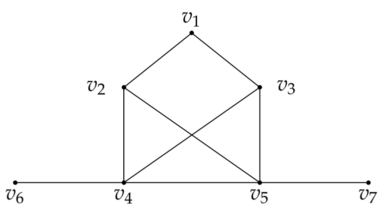

Figure 2 Graph G

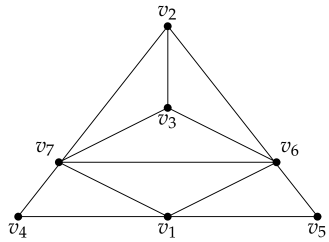

Figure 3 Graph $\overline{G}$

Proposition 4.5. *Let G be a graph satisfying the N-condition with $3 \leq \omega(G) < \infty$. Then G is strongly complemented with $\omega(G) = \omega(\overline{G})$ if and only if G is a complete graph with each vertex adjacent to an end vertex.*

Proof. ($\Longleftarrow$) If G is a complete graph with each vertex adjacent to an end vertex, denote the complete graph by S, which is the unique maximum clique of G. For each $x \in V(S)$, let $\overline{x}$ be the end vertex adjacent to x. Clearly, $V(G) = V(S) \cup \{\overline{x} \mid x \in V(S)\}$. It is easy to see that for each $x \in V(S)$, $\overline{x}$ is the unique complement of x, and vice versa. So, G is strongly complemented. By Corollary 4.3, $\omega(G) \leq \omega(\overline{G})$. Since $\omega(G) \geq 3$, $V(S) \nsubseteq \{x, y\} = N(\overline{x}) \cup N(\overline{y})$ for each pair $x, y \in V(S)$. Hence $\{\overline{x} \mid x \in V(S)\}$ induces a discrete subgraph of G by Lemma 3.5. For each $z \in V(S)$, there exists a $\overline{z} \in \{\overline{x} \mid x \in V(S)\}$, such that $z \sim \overline{z}$, hence $\{\overline{x} \mid x \in V(S)\}$ induces a maximum discrete subgraph of G, i.e., $\{\overline{x} \mid x \in V(S)\}$ induces a maximum clique in $\overline{G}$.

($\Longrightarrow$) Note that G has no isolated vertex since G is complemented. If G satisfies the N-condition, by Corollary 3.6, G is a blow-up of a pre-atomic graph. If G is strongly complemented, then clearly G is a pre-atomic graph. Let S be the unique maximum clique of G. Then for each $x \in V(S)$, there exists an unique $\overline{x}$ such that $\overline{x} \perp x$. Clearly, $V(S) \cap N(\overline{x}) = \{x\}$ for each $x \in V(S)$. Let M be the graph induced by $V(S) \cup \{\overline{x} \mid x \in V(S)\}$. In the following, we will show that $G = M$. If there exists a $y \in V(G) \setminus V(M)$, then consider the following two cases:

Case 1: $y \sim \overline{x}$ for some $x \in V(S)$. By Lemma 3.5, $V(S) \subseteq N(y) \cup N(\overline{x})$, and hence $V(S) \setminus \{x\} \subseteq N(y)$. Since G is a pre-atomic graph, $y = x \in V(S)$, a contradiction.

Case 2: $y \nsim \overline{x}$ for each $\{\overline{x} \mid x \in V(S)\}$. The subgraph induced by $\{y\} \cup \{\overline{x} \mid x \in V(S)\}$ is a discrete subgraph properly containing $\{\overline{x} \mid x \in V(S)\}$, it contradicts to $\omega(G) = \omega(\overline{G})$. □

5. Characterizing a blow-up of a Boolean graph by the conditions M, N or N^*

In this section, we will give a new characterization of Boolean graphs and blow-ups of a Boolean graph.

Theorem 5.1. *Let G be a graph with a maximum clique S. Then G is a blow-up of a Boolean graph if and only if G is a complemented graph and satisfies conditions M and N.*

Proof. ($\Longrightarrow$) By Theorem 2.6, for each pair of x, y in a blow-up of a Boolean graph, $x \in N(y)$ if and only if $V(S) \subseteq N(x) \cup N(y)$. By Lemma 3.10, it suffices to show that G is complemented and satisfies the M-condition. Let G be a blow-up of B_S. By the definitions of complemented graph and M-condition, what is involved is nothing but relations of neighbourhoods of vertices, so it is sufficient to check that B_S is complemented and satisfies the M-condition. Note that $V(B_S) = 2^X \setminus \{X, \emptyset\}$ for some finite or infinite set X, and B_S has a unique maximum clique $S = \{\{t\} \mid t \in X\}$.

For each nontrivial sub-clique A of S, denote $v_A = \{i \mid \{i\} \in V(A)\} \in V(B_S)$. Clearly, each vertex of B_S can be written as v_A for some sub-clique A, and

$$v_{S\setminus A} = \{i \mid \{i\} \in V(S) \setminus V(A)\}$$

is a complement of v_A. So, B_S is a complemented graph. In the following, we will show that B_S satisfies the M-condition.

Let Δ be an induced discrete subgraph of B_S. If $V(S) \nsubseteq \bigcup_{v_A \in V(\Delta)} N(v_A)$, since

$$N(v_A) = \{v_C \mid V(C) \subseteq V(S) \setminus V(A)\}$$

holds for each $v_A \in \Delta$,

$$V(S) \neq \bigcup_{v_A \in \Delta} (V(S) \setminus V(A)) = V(S) \setminus \bigcap_{v_A \in \Delta} V(A).$$

Hence $\cap_{v_A \in \Delta} V(A) \neq \emptyset$. Denote E the subgraph induced by $\cap_{v_A \in \Delta} V(A)$, then $v_E \in V(B_S)$ and

$$N(v_E) = \{v_C \mid V(C) \subseteq V(S) \setminus \bigcap_{v_A \in \Delta} V(A)\} = \{v_C \mid V(C) \subseteq \bigcup_{v_A \in \Delta} (V(S) \setminus V(A))\}.$$

Clearly, $\cup_{v_A \in \Delta} N(v_A) \subseteq N(v_E)$ and, for each $\{i\} \in V(S)$, the following implications holds

$$\begin{aligned} \{i\} \in N(v_E) &\Longleftrightarrow \{i\} \notin \bigcap_{v_A \in \Delta} V(A) \Longleftrightarrow \exists v_B \in \Delta \text{ such that } \{i\} \notin V(B) \\ &\Longleftrightarrow \exists v_B \in \Delta \text{ such that } \{i\} \in N(v_B) \Longleftrightarrow \{i\} \in \bigcup_{v_A \in \Delta} N(v_A). \end{aligned}$$

So, B_S satisfies the M-condition.

($\Longleftarrow$) If G is complemented, then there is no isolated vertex in G. Because G satisfies the N-condition without isolated vertex, so G is a compact graph. Hence G is a blow-up of a pre-atomic graph by Corollary 3.6. Next, we consider the following two cases:

case 1: $|V(S)| = 2$.

Since a pre-atomic graph G with $\omega(G) = 2$ is B_2, the proof is completed.

case 2: $|V(S)| \geq 3$ or $|V(S)| = \infty$.

By Theorem 2.6 and Proposition 2.9, it suffices to prove that G satisfies the latter part of the condition (1) in Theorem 2.6. In fact, for each nontrivial $V(B) \subseteq V(S)$, since G is a complemented graph, for each $v_i \in V(B)$, there exists a $\overline{v_i} \in V(G)$ such that $\overline{v_i} \perp v_i$. Let A be the subgraph induced by $\{\overline{v_i} \mid v_i \in V(B)\}$. We claim that A is discrete subgraph of G. It is easy to see that $V(S) \cap N(\overline{v_i}) = \{v_i\}$. Otherwise, if there exists $v_j \in V(S) \cap N(\overline{v_i})$ and $v_j \neq v_i$, then $v_j \sim v_i$ and $v_j \sim \overline{v_i}$, a contradiction. Hence $V(S) \nsubseteq N(\overline{v_i}) \cup N(\overline{v_j})$ for each pair $\overline{v_i}, \overline{v_j} \in V(A)$. By Lemma 3.5, since G satisfies the N-condition, so $\overline{v_i} \nsim \overline{v_j}$ for each pair $\overline{v_i}, \overline{v_j} \in V(A)$. Clearly, $V(S) \cap (\cup_{\overline{v_i} \in V(A)} N(\overline{v_i})) = V(B)$ implies $V(S) \nsubseteq \cup_{\overline{v_i} \in V(A)} N(\overline{v_i})$. Since G satisfies the M-condition, by Definition 3.8, there exists $u_A \in V(G)$ such that

$$V(S) \cap N(u_A) = V(S) \cap (\bigcup_{\overline{v_i} \in V(A)} N(\overline{v_i})) = V(B).$$

This completes the proof. □

By Theorem 5.1 and Proposition 3.11, it is easy to obtain the following corollary.

Corollary 5.2. *Let G be a graph with a finite maximum clique S. Then G is a blow-up of a Boolean graph if and only if G is a complemented graph and satisfies the N^*-condition.*

Theorem 5.3. *Let G be a graph with a maximum clique S. Then G is a Boolean graph if and only if G is strongly complemented and satisfies conditions M and N.*

Proof. ($\Longrightarrow$) By Theorem 5.1, it is clear.

($\Longleftarrow$) Assume that G is strongly complemented and satisfies conditions M and N. By Theorem 5.1, G is a blow-up of a Boolean graph, thus satisfies conditions of Theorem 2.6. In the following, it suffices to show that G satisfies condition (2) in Theorem 2.2, i.e., G is uniquely N-determined. We claim that it is true. Assume to the contrary that there are distinct vertices $x, y \in V(G)$ such that $N(x) = N(y)$. Since G is complemented, there exists a vertex $z \in V(G)$, such that $z \perp x$, i.e., $z \in N(x)$ and $N(x) \cap N(z) = \emptyset$. Note that $N(x) = N(y)$, so $z \perp y$, contradicting the strongly complemented assumption on G. □

By Theorem 5.3 and Proposition 3.11, it is easy to check the following corollary.

Corollary 5.4. *Let G be a graph with a finite maximum clique S. Then G is a Boolean graph if and only if G is strongly complemented and satisfies the N^*-condition.*

6. Application to co-maximal ideal graph $\mathcal{C}(R)$

In this section, by applying the previous characterizations to the co-maximal ideal graph $\mathcal{C}(R)$, we have got several interesting new results, and we provide an alternative way for proving the main theorem of [21]. we assume that the ring R appeared in the following is a commutative ring with identity. Recall that the co-maximal ideal graph $\mathcal{C}(R)$ of a ring R is a connected graph, with vertex set

$$\{I \mid I \text{ is a proper ideal of } R, \text{ and } I \nsubseteq J(R)\},$$

where I is adjacent to J if and only if $I + J = R$. Clearly, S is a maximum clique of $\mathcal{C}(R)$, which is induced by $Max(R)$ in $\mathcal{C}(R)$. See also [21, 22].

Proposition 6.1. *$\mathcal{C}(R)$ satisfies conditions N and M.*

Proof. For each pair of ideals I_1, I_2, if $I_1 \notin N(I_2)$, i.e., $I_1 + I_2 \neq R$, then there exists a maximal ideal J, such that $I_1 + I_2 \subseteq J$. Clearly, $N(I_1) \cup N(I_2) \subseteq N(J)$. Hence $\mathcal{C}(R)$ satisfies the N-conditions.

Clearly, S, induced by $Max(R)$, is a maximum clique of $\mathcal{C}(R)$. If $\{I_i \mid i \in \Gamma\}$ induces a discrete subgraph in $\mathcal{C}(R)$, and $V(S) \nsubseteq \cup_{i\in\Gamma} N(I_i)$. Then there exists $L \in V(S)$, such that for each $i \in \Gamma$, $I_i \subseteq L$. Hence $K = \{x \in \sum_{i\in A} I_i \mid A \subseteq \Gamma \text{ and } |A| < \infty\} \neq R$ is an ideal such that $\cup_{i\in\Gamma} N(I_i) \subseteq N(K)$ and $V(S) \cap (\cup_{i\in\Gamma} N(I_i)) = V(S) \cap N(K)$. In fact, it suffices to check that for each $J \in V(S)$, $J \in N(K)$ implies $J \in \cup_{i\in\Gamma} N(I_i)$. Actually, if $J \in N(K)$, then there exists $x \in K$, such that $x \notin J$ and $x \in \sum_{i\in A} I_i$ for some finite subset A of Γ. Without loss of generality, assume that $x = x_1 + x_2 + \cdots + x_n \in I_1 + I_2 + \cdots + I_n$, then there exists $x_i \in I_i$ such that $x_i \notin J$. Hence $J \in N(I_i)$ and thus, $\mathcal{C}(R)$ satisfies the condition M. □

Proposition 6.2. *If $\omega(\mathcal{C}(R)) < \infty$, then $\mathcal{C}(R)$ is complemented.*

Proof. If $\omega(C(R)) = n < \infty$, then $|Max(R)| = n$. Let $Max(R) = \{M_1, M_2, \cdots, M_n\}$. For each ideal I, let $A_I = \{i \,|\, I \nsubseteq M_i\} \subsetneq [n]$. We show that $J = \cap_{i \in A_I} M_i$ is a complement of I in $C(R)$. In fact, if $I + J \neq R$, then there exists $M_j \in Max(R)$, such that $I + J \subseteq M_j$. Then $j \notin A_I$ since $I \subseteq M_j$. Set $x = \prod_{i \in A_I} x_i$, where $x_i \in M_i \setminus M_j$ for each $i \in A_I$. Note that on the one hand $J = \cap_{i \in A_I} M_i \subseteq M_j$, on the other hand there exists $x \in \cap_{i \in A_I} M_i \setminus M_j$, a contradiction. So, $J \in N(I)$. In the following, we will show that there is no ideal of R which is adjacent to both I and J in $C(R)$. It follows by noting that every ideal adjacent to I can not be contained in M_i for each $i \in [n] \setminus A_I$, and every ideal adjacent to J can not be contained in M_i for each $i \in A_I$. □

In [21], the authors show that $C(R)$ may not be a blow-up of B_∞ when $\omega(C(R)) = \infty$. To some extent, it is because $C(R)$ may be not complemented in this case.

By Theorem 5.1, Proposition 6.1 and Proposition 6.2, it is easy to deduce the following corollary. Actually, the equivalence of (1) to (3) is the main theorem of [21].

Corollary 6.3. ([21, Theorem 3.5]) *For a commutative ring R and its co-maximal ideal graph $C(R)$, the following statements are equivalent:*
(1) $|Max(R)| = n < \infty$;
(2) $\omega(C(R)) = n < \infty$;
(3) $C(R)$ *is a blow-up of a finite Boolean graph* B_n;
(4) $C(R)$ *is a blow-up of a finite pre-atomic graph* A_n.

In [7], we will use the characterizations to study annihilating ideal graphs of rings which are blow-ups of Boolean graphs (complemented graphs, respectively).

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